

A new general relativistic clock effect for counter-rotating test particles in the gravitoelectric field of a non-rotating body

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Abstract

It is known that, in the weak-field and slow-motion approximation of general relativity, the stationary component of the gravitational field of a rotating body, proportional to its angular momentum, is able to discriminate between the opposite directions of motion of a pair of counter-orbiting test particles moving along geometrically identical paths: it is the so-called gravitomagnetic clock effect. In this paper, we show that, to the first post-Newtonian level, there is also a gravitoelectric clock effect induced by the static component of the gravitational field of a non-rotating body. Indeed, the difference of the draconitic periods of two test particles moving in opposite directions along identical quasi-circular orbits turns out to be non-vanishing. It depends on the shape, the size and the pericenter location of the orbit, but it is independent of the initial position along it. On the contrary, the gravitoelectric corrections to the anomalistic periods turn out to be the same for both prograde and retrograde orbital motions. For a hypothetical pair of Earth satellites in Supertundra-like orbits, the gravitoelectric draconitic clock effect could be as large as $46 \mu\text{s}$, while a Juno-like orbital configuration around Jupiter would allow to reach about 0.01 s.

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1 Introduction

At the post-Newtonian level, it has always been known so far that it is the angular momentum of a rotating body, acting as source of the stationary component of its gravitational field known as gravitomagnetic, that discriminates between opposite directions of motion of a pair of counter-revolving test particles moving along otherwise geometrically identical orbits. Indeed, it turns out that several characteristic temporal intervals of the orbital motions differ from each other in such a way that nonvanishing differences between such characteristic times occur giving rise to different versions of the so-called gravitomagnetic clock effect [1]. On the contrary, both the Newtonian and the post-Newtonian gravitoelectric components of the body's gravitational field are known not to allow for such temporal asymmetry, thus allowing, at least in principle, for the possibility of singling out just the relativistic time lapse.

In this paper, we will show that, in fact, also the post-Newtonian static component of the gravitational field of a non-rotating body does induce a characteristic time shift for a pair of test particles traveling along opposite directions on identical elliptical orbits. It occurs if the draconitic period, which is the temporal interval between two successive passages at the ascending node, is considered. Instead, the anomalistic period, which refers to two consecutive passages at the pericenter, does not exhibit such a peculiarity.

The paper is organized as follows. Section 2 explains how to perturbatively calculate both the draconitic and the anomalistic periods of a test particle when a small disturbing acceleration is present. In Section 3, the effect of the post-Newtonian gravitoelectric field on the draconitic period is analytically worked out. Section 4 is devoted to the calculation of the gravitoelectric correction to the anomalistic period. In Section 5, we present our conclusions.

Notations

Here, basic notations and definitions used in the text are presented [2,3].

G : Newtonian constant of gravitation

c : speed of light in vacuum

M : mass of the primary

$\mu = GM$: gravitational parameter of the primary

a : semimajor axis

$T_0 = 2\pi n_b^{-1}$: Keplerian orbital period

e : eccentricity

$p = a(1 - e^2)$: semilatus rectum

I : inclination of the orbital plane

Ω : longitude of the ascending node

ω : argument of pericenter

f : true anomaly

$u = \omega + f$: argument of latitude

$q = e \cos \omega$: nonsingular orbital element q

$k = e \sin \omega$: nonsingular orbital element k

\mathbf{A} : disturbing acceleration

A_R, A_T, A_N : radial, transverse and normal components of \mathbf{A}

T_{dr} : draconitic period

T_{an} : anomalistic period

2 Characteristic orbital periods for a perturbed orbit

2.1 The draconitic period

The draconitic period T_{dr} of a perturbed orbit is the time interval between two successive instants when the real position of the test particle coincides with the ascending node position on the corresponding osculating orbit. It can be measured as proposed in [4–6]. It can be calculated as [7]

$$T_{\text{dr}} = \int_0^{2\pi} \left(\frac{dt}{du} \right) du. \quad (1)$$

From the definition of the argument of latitude, it is

$$\frac{du}{dt} = \frac{d\omega}{dt} + \frac{df}{dt}, \quad (2)$$

where [2, 8]

$$\frac{df}{dt} = \frac{\sqrt{\mu p}}{r^2} - \frac{d\omega}{dt} - \cos I \frac{d\Omega}{dt}. \quad (3)$$

Thus, it can be written

$$\frac{du}{dt} = \frac{\sqrt{\mu p}}{r^2} \left(1 - \frac{r^2 \cos I}{\sqrt{\mu p}} \frac{d\Omega}{dt} \right), \quad (4)$$

so that

$$\frac{dt}{du} = \frac{r^2 \alpha}{\sqrt{\mu p}}, \quad (5)$$

where we defined

$$\alpha \doteq \frac{1}{1 - \frac{r^2 \cos I}{\sqrt{\mu p}} \frac{d\Omega}{dt}}. \quad (6)$$

To the first order in the disturbing acceleration, eq. (5) can be expanded as

$$\frac{dt}{du} \simeq \frac{r^2}{\sqrt{\mu p}} + \frac{r^4 \cos I}{\mu p} \frac{d\Omega}{dt}. \quad (7)$$

In calculating eq. (1) through eq. (7), the second term of eq. (7), with

$$\frac{d\Omega}{dt} = \frac{r \sin u A_N}{\sqrt{\mu p} \sin I} \quad (8)$$

evaluated onto the unperturbed Keplerian ellipse, yields the direct correction to the draconitic period. Actually, a further contribution arises also from the first term in eq. (7) when the instantaneous shifts of the orbital elements $\{\xi\}$ entering it are properly taken into account. In the notation of [9], it can be dubbed as “indirect”. More specifically, from

$$r = \frac{p}{1 + q \cos u + k \sin u}, \quad (9)$$

by posing

$$F(p, q, k, u) \doteq \frac{r^2}{\sqrt{\mu p}}, \quad (10)$$

it is

$$F = F_0 + \Delta F = F_0 + \sum_{\xi} \left(\frac{\partial F}{\partial \xi} \right)_0 \Delta \xi(u_0, u), \quad (11)$$

where the subscript “0” refers to the unperturbed, Keplerian ellipse.

Thus, we have

$$T_{\text{dr}} = T_0 + T_{\text{dr}}^{(A)}, \quad (12)$$

where

$$T_{\text{dr}}^{(A)} = I_1 + I_2 + I_3 + I_4, \quad (13)$$

with

$$I_1 = \frac{3}{2} \sqrt{\frac{p}{\mu}} \int_0^{2\pi} \frac{\Delta p}{(1 + q \cos u + k \sin u)^2} du, \quad (14)$$

$$I_2 = -2 \sqrt{\frac{p^3}{\mu}} \int_0^{2\pi} \frac{\Delta q \cos u}{(1 + q \cos u + k \sin u)^3} du, \quad (15)$$

$$I_3 = -2 \sqrt{\frac{p^3}{\mu}} \int_0^{2\pi} \frac{\Delta k \sin u}{(1 + q \cos u + k \sin u)^3} du, \quad (16)$$

$$I_4 = \int_0^{2\pi} \frac{r^4 \cos I}{\mu p} \frac{d\Omega}{dt} du. \quad (17)$$

In eq. (14)-eq. (17), the instantaneous shifts of p, q, k are to be calculated as [7]

$$\Delta p = \int_{u_0}^u \left(\frac{dp}{du'} \right) du', \quad (18)$$

$$\Delta q = \int_{u_0}^u \left(\frac{dq}{du'} \right) du', \quad (19)$$

$$\Delta k = \int_{u_0}^u \left(\frac{dk}{du'} \right) du', \quad (20)$$

by using the analytical expressions [7, 10]

$$\frac{dp}{du} = \frac{2r^3\alpha A_T}{\mu}, \quad (21)$$

$$\frac{dq}{du} = \frac{r^3\alpha k \sin u \cot I A_N}{\mu p} + \frac{r^2\alpha \left[\frac{r}{p} (q + \cos u) + \cos u \right] A_T}{\mu} + \frac{r^2\alpha \sin u A_R}{\mu}, \quad (22)$$

$$\frac{dk}{du} = -\frac{r^3\alpha q \sin u \cot I A_N}{\mu p} + \frac{r^2\alpha \left[\frac{r}{p} (k + \sin u) + \sin u \right] A_T}{\mu} - \frac{r^2\alpha \cos u A_R}{\mu}. \quad (23)$$

to the first order in the disturbing acceleration, i.e. by setting $\alpha = 1$. It is intended that eq. (14)-eq. (23) are evaluated onto the unperturbed Keplerian ellipse.

2.2 The anomalistic period

The anomalistic period T_{an} of a perturbed orbit is the time interval between two successive instants when the real position of the test particle coincides with the pericenter position on the corresponding orbit. It can be calculated as [11, 12]

$$T_{\text{an}} = \int_0^{2\pi} \left(\frac{dt}{df} \right) df, \quad (24)$$

in which

$$\frac{dt}{df} = \frac{r^2\beta}{\sqrt{\mu p}}, \quad (25)$$

$$\beta = \frac{1}{1 - \frac{r^2}{\sqrt{\mu p}} \left(\frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} \right)}. \quad (26)$$

By proceeding as in Section 2.1, it is possible to write

$$T_{\text{an}}^{(A)} = J_1 + J_2 + J_3, \quad (27)$$

with [11, 12]

$$J_1 = \frac{3}{2} \sqrt{\frac{p}{\mu}} \int_0^{2\pi} \frac{\Delta p}{(1 + e \cos f)^2} df, \quad (28)$$

$$J_2 = -2 \sqrt{\frac{p^3}{\mu}} \int_0^{2\pi} \frac{\Delta e \cos f}{(1 + e \cos f)^3} df, \quad (29)$$

$$J_3 = \int_0^{2\pi} \frac{r^4}{\mu p} \left(\frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} \right) df. \quad (30)$$

The integrals of eq. (28)-eq. (30) can be worked out by means of [12, 13]

$$\frac{dp}{df} = \frac{2r^3 \beta A_T}{\mu}, \quad (31)$$

$$\frac{de}{df} = \frac{r^2 \beta}{\mu} \left[\sin f A_R + \left(1 + \frac{r}{p} \right) \cos f A_T + e \left(\frac{r}{p} \right) A_T \right], \quad (32)$$

$$\frac{d\Omega}{df} = \frac{r^3 \beta \sin(\omega + f) A_N}{\mu p \sin I}, \quad (33)$$

$$\frac{d\omega}{df} = \frac{r^2 \beta}{\mu} \left[-\frac{\cos f A_R}{e} + \left(1 + \frac{r}{p} \right) \frac{\sin f A_T}{e} - \left(\frac{r}{p} \right) \cot I \sin(\omega + f) A_N \right], \quad (34)$$

evaluated onto the unperturbed Keplerian ellipse and for $\beta = 1$.

3 The gravitoelectric draconitic period and the time shift for counter-orbiting satellites

In the case of the 1PN gravitoelectric acceleration due to a static mass M , its radial, transverse and normal components are [14]

$$A_R^{(\text{GE})} = \frac{\mu^2 (1 + e \cos f)^2 (3 + e^2 + 2e \cos f - 2e^2 \cos 2f)}{c^2 p^3}, \quad (35)$$

$$A_T^{(\text{GE})} = \frac{4\mu^2 (1 + e \cos f)^3 e \sin f}{c^2 p^3}, \quad (36)$$

$$A_N^{(\text{GE})} = 0. \quad (37)$$

By inserting eq. (35)-eq. (37) into eq. (18)-eq. (20) with eq. (21)-eq. (23), it can be obtained

$$\Delta p = \frac{8e\mu [\cos(u_0 - \omega) - \cos(u - \omega)]}{c^2}, \quad (38)$$

$$\begin{aligned} \Delta q = & \frac{\mu}{c^2 p} \{ 3(1 + e^2)(\cos u_0 - \cos u) + \\ & + e[4e \cos(u_0 - 2\omega) - 4e \cos(u - 2\omega) + \\ & + 5 \sin(u - u_0) \sin(u + u_0 - \omega) - 3(u - u_0) \sin \omega] \}, \end{aligned} \quad (39)$$

$$\begin{aligned} \Delta k = & -\frac{\mu}{c^2 p} \{ 3(1 + e^2)(\sin u - \sin u_0) + \\ & + e[4e \sin(u_0 - 2\omega) - 4e \sin(u - 2\omega) + \\ & + 5 \sin(u - u_0) \cos(u + u_0 - \omega) - 3(u - u_0) \cos \omega] \}. \end{aligned} \quad (40)$$

We are now ready to calculate the gravitoelectric correction $T_{\text{dr}}^{(\text{GE})}$ to the draconitic period. From eq. (8) and eq. (37), it turns out that $I_4 = 0$. In order to calculate eq. (14)-eq. (16), a power expansion in e is required. To the first order in e , one gets

$$T_{\text{dr}}^{(\text{GE})} = \frac{6\pi\sqrt{\mu a}(2 + 7e \cos f_0 + 2e \cos \omega)}{c^2} + \mathcal{O}(e^2). \quad (41)$$

It can be noticed that eq. (41), which vanishes in the limit $\mu \rightarrow 0$, does depend on the size of the orbit. Moreover, it depends also on the initial position along it.

The motion of a counter-revolving test particle, labelled by the superscript “(-)”, along an otherwise geometrically identical orbit is characterized

by

$$u^{(-)} = \pi - u, \quad (42)$$

$$\omega^{(-)} = \pi - \omega, \quad (43)$$

$$f^{(-)} = -f \quad (44)$$

since both u and ω are reckoned from the ascending node which corresponds to the descending node of the opposite direction of motion. It is interesting to notice that, according to eq. (35)-eq. (36), while the radial component of the gravitoelectric acceleration remains unchanged, the transverse one, proportional to the eccentricity, gets reversed. As such, by repeating the previous calculation for the reversed direction of motion, one has

$$T_{\text{dr}}^{(-)} = \frac{6\pi\sqrt{\mu a} (2 + 7e \cos f_0 - 2e \cos \omega)}{c^2} + \mathcal{O}(e^2). \quad (45)$$

It yields a generally nonvanishing difference of the gravitoelectric draconitic periods for quasi-circular orbits

$$\Delta T_{\text{dr}}^{(\text{GE})} = \frac{24\pi e \sqrt{\mu a} \cos \omega}{c^2} + \mathcal{O}(e^2). \quad (46)$$

Note that eq. (46) is independent of the initial position of the test particles.

For a hypothetical pair of Earth satellites with $a = 42163.191$ km, $e = 0.423$, which correspond to the Supertundra orbits [15], the shift of eq. (46) can be as large as

$$\frac{\Delta T_{\text{dr}}^{(\text{GE})}}{\cos \omega} = 46 \text{ } \mu\text{s}. \quad (47)$$

A Juno-like orbital geometry around Jupiter would yield a time shift as large as 0.01 s.

4 The gravitoelectric anomalistic period

The calculation of the gravitoelectric correction to the anomalistic period can be performed without recurring to a power expansion in e of eq. (28)-eq. (30). The resulting exact expression is

$$T_{\text{an}}^{(\text{GE})} = \frac{3\pi\sqrt{\mu a} [6 + 7e^2 + 2e^4 + 2e(7 + 3e^2) \cos f_0 + 5e^2 \cos 2f_0]}{c^2 (1 - e^2)^2}. \quad (48)$$

It depends on the initial position of the test particle along the orbit.

It turns out that eq. (48) does not change if the direction of motion is reversed. Thus, no gravitoelectric anomalistic clock effect occurs.

5 Conclusions

We have demonstrated that also the post-Newtonian gravitoelectric component of the static gravitational field of a non-rotating body can discriminate between opposite directions of motion for a pair of counter-revolving test particles moving along identical quasi-circular orbits, provided that their draconitic periods are considered. Instead, the anomalistic periods of prograde and retrograde motions are identical.

Such a gravitoelectric draconitic clock effect may be as large as $46 \mu\text{s}$ around the Earth if the semimajor axis and the eccentricity of the Super-tundra orbital configuration are assumed. Instead, a Juno-like orbit around Jupiter would allow to reach up to 0.01 s.

Careful investigations about the impact of the unavoidably partial cancellation of the classical contributions to the draconitic periods, which, in principle, are identical for both the directions of motion, will be the subject of further studies.

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